PHYS 331 - Oct. 23, 2023

Last Time:

Trans. line reflection coefficient $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$

Useful results:

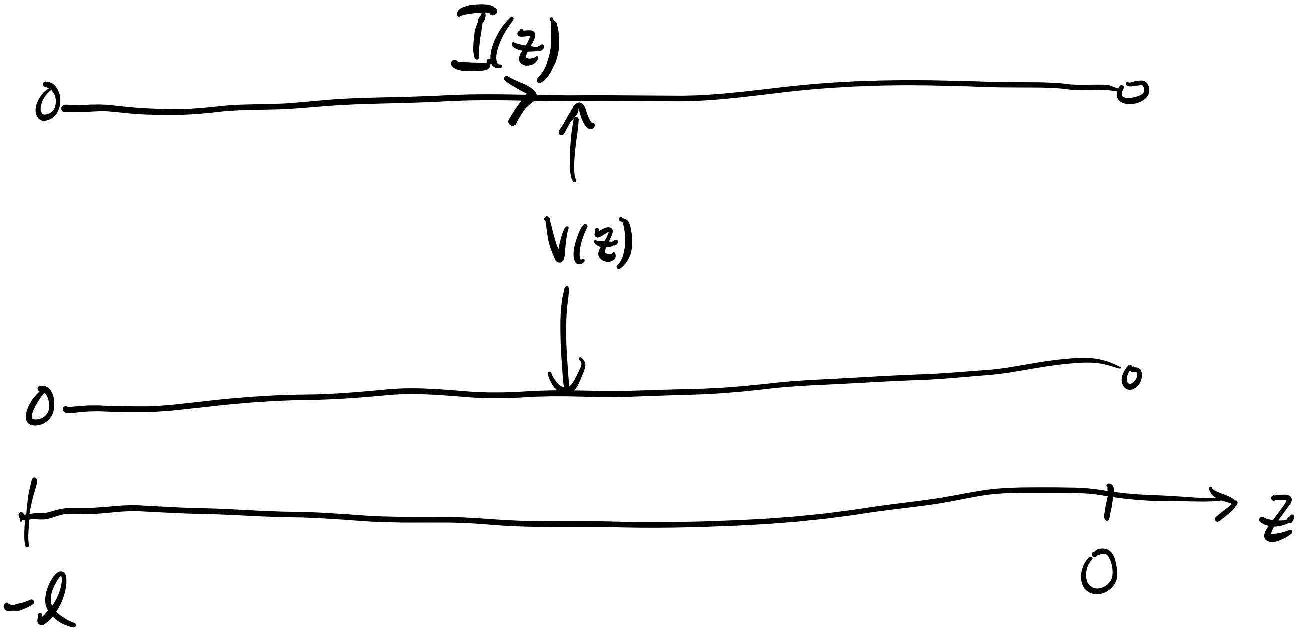
volt. amp. $V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$

current amp. $I(z) = \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$

$$Z_0 = \sqrt{\frac{L_l}{C_l}}$$

Today:
- when are trans. line effects important?
- impedance looking into a trans. line Z_{in}.
- demo in SCI 239.

When are trans. line effects important?



$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

$$I(z) = \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{L_l C_l} \qquad Z_0 = \sqrt{\frac{L_l}{C_l}}$$

The coaxial cables used in the lab are called RG-58 & designed s.t.

$$\left.\begin{aligned} C_\ell &\approx 80\text{pF/m} \\ L_\ell &\simeq 0.20\,\mu\text{H/m} \end{aligned}\right\} \quad Z_0 = \sqrt{\frac{L_\ell}{C_\ell}} \approx 50\Omega.$$

$\beta = \omega\sqrt{L_\ell C_\ell}$ let's take $\omega = 2\pi(10\text{kHz})$

$\approx 2.5\times10^{-4}\ \text{m}^{-1} \ll 1\ \text{m}^{-1}$

If we take $z \simeq 1\text{m}$ (typical)

then $\beta z \simeq 2.5\times10^{-4} \ll 1$

$$\therefore\ V(z) = V_+ \overset{1}{\cancel{e^{-j\beta z}}} + V_- \overset{1}{\cancel{e^{j\beta z}}} \quad \text{for } \beta z \ll 1$$

$$V(z) \simeq V_+ + V_- \quad (\text{const}).$$

for current amp., would find

$$I(z) \simeq \frac{1}{Z_0}\left[V_+ - V_-\right] \ (\text{const})$$

When $\beta z \ll 1$, current & volt. amp is uniform along length of trans. line & we can ignore trans. line effects.

Recall $\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta}$

using $\beta = 2.5\times10^{-4}\ m^{-1}$, we get

$\lambda = 25\ km$!

When λ much longer than the trans. line length ℓ, we can ignore these effects.

If, on the other hand, we take

$$\omega = 2\pi(\underbrace{3\,GHz}_{3\times10^{9}\,Hz})$$

then $B = \omega\sqrt{L_\ell C_\ell} \approx 75\ m^{-1}$

In this case, for $z \simeq 1$ m, the product βz not small.

Cannot approx. $e^{\pm j\beta z} \simeq 1$.

Now $V(z)$ & $I(z)$ do vary along length line ⇒ trans. line effects are important.

$$\lambda = \frac{2\pi}{\beta} \simeq 8\text{cm} \ll l.$$

Impedance at an arbitrary pos. z along transmission line.

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

$$I(z) = \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

at an arbitrary pt. along trans. line, the impedance $Z(z)$

impedance $\nearrow$ z position

$$* \quad Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_+ e^{-j\beta z} + V_- e^{j\beta z}}{V_+ e^{-j\beta z} - V_- e^{j\beta z}}$$

Recall that $V_- = \Gamma V_+$

$$Z(z) = Z_0 \frac{\cancel{V_+}\left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)}{\cancel{V_+}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)}$$

Sub $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{j\beta z}}{e^{-j\beta z} - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{j\beta z}}$$

mult. by $\frac{Z_L + Z_0}{Z_L + Z_0}$

$$Z(z) = Z_0 \frac{(Z_L + Z_0)e^{-j\beta z} + (Z_L - Z_0)e^{j\beta z}}{(Z_L + Z_0)e^{-j\beta z} - (Z_L - Z_0)e^{j\beta z}}$$

collect like terms of Z_L & Z_0.

$$Z(z) = Z_0 \frac{Z_L\left(e^{-j\beta z} + e^{j\beta z}\right) + Z_0\left(e^{-j\beta z} - e^{j\beta z}\right)}{Z_L\left(e^{-j\beta z} - e^{j\beta z}\right) + Z_0\left(e^{-j\beta z} + e^{j\beta z}\right)}$$

Identities: $e^{+j\beta z} + e^{-j\beta z} = 2\cos\beta z$

$$e^{j\beta z} - e^{-j\beta z} = 2j\sin\beta z$$

$$Z(z) = Z_0 \frac{\cancel{2}Z_L\cos\beta z - \cancel{2}jZ_0\sin\beta z}{-\cancel{2}jZ_L\sin\beta z + \cancel{2}Z_0\cos\beta z}$$

divide by $\cos\beta z$

$$Z(z) = Z_0 \frac{Z_L - jZ_0\tan\beta z}{Z_0 - jZ_L\tan\beta z}$$

To find the impedance looking into trans. line from its start, sub in $z = -l$. Recall that $\tan(-\beta l)$

$= - \tan \beta l$

$$\boxed{Z_{in} \equiv Z(-l) = Z_0 \frac{Z_L + j Z_0 \tan \beta l}{Z_0 + j Z_L \tan \beta l}}$$

The impedance of a length of trans. line depends on :

- load impedance Z_L at the end.
- geometry of trans. line which sets value of Z_0
- length l of trans. line
- freq. ω of signal through $\beta = \omega\sqrt{LC}$

Special cases:

① $\beta l = n\pi$ → select l s.t. we satisfy $\beta l = n\pi$ where n integer.

$$\frac{2\cancel{\pi}}{\lambda} l = n\cancel{\pi} \Rightarrow l = n\left(\frac{\lambda}{2}\right)$$

Select l to be integer mult. of $\lambda/2$

$$\tan \beta l = \tan n\pi = 0.$$

$$Z_{in} = \cancel{Z_0} \frac{Z_L}{\cancel{Z_0}} \Rightarrow \boxed{Z_{in} = Z_L}$$

like trans. line is not there.

② $\beta l = n'\left(\frac{\pi}{2}\right)$ n': odd # integer.

$$\frac{2\cancel{\pi}}{\lambda} l = n'\left(\frac{\cancel{\pi}}{2}\right)$$

$$l = n'\left(\frac{\lambda}{4}\right)$$

odd. mult. of quarter wavelength

$$\tan(\beta l) = \tan\left(n' \frac{\pi}{2}\right) \rightarrow \pm\infty$$

In this case

$$Z_{in} \approx Z_0 \frac{\cancel{j} Z_0 \cancel{\tan\beta l}}{\cancel{j} Z_L \cancel{\tan\beta l}}$$

$$\boxed{Z_{in} = \frac{Z_0^2}{Z_L}}$$

"transform the impedance".

③ $Z_L = Z_0$ (impedance matching condition).

$$Z_{in} = Z_0 \frac{\cancel{\cancelto{Z_0}{Z_L} + jZ_0 \tan\beta l}}{\cancel{Z_0 + j\cancelto{Z_0}{Z_L} \tan\beta l}}$$

$$\boxed{\therefore Z_{in} = Z_0}$$